1 Rules of the Game

Topology is a game where all objects you want to create are made of an infinitely thin material that can stretch, shrink, bend, and ripple. On a geometry class, everything you create cannot be changed - if you draw a triangle with sides 3, 4 and 5, it remains such a triangle until the the end of the universe. However, topology does not care about the sizes! From the point of view of topology, all triangles are the same! Moreover, triangles and circles are the same, squares and circles are the same. We can continue this forever - all polygons are the same as a circle. Everything written above seems strange at first. But let's scale it up one dimension to clarify! Let's take a piece of play dough and form a sphere; then, make a cube from that piece of play dough; then, make a tetrahedron. We have just proved that any sphere is the same as any three-dimensional solid. (Caution! With no holes!) If you want to prove two objects are the same topologically, you take your first object and bend it, stretch it, shrink it, not ripping anywhere, transforming it into the other object.

Practice

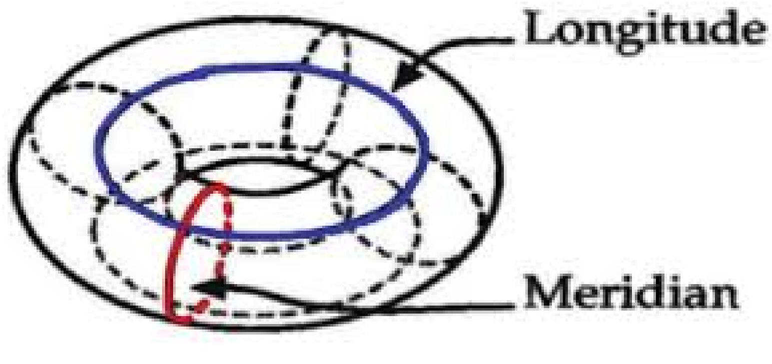
- Draw any kind of curved enclosed line and show that it is homeomorphic to (the same as) a circle.
- Prove that a cylinder with no top is homeomorphic to a circle.
- Prove that an n-gon with no contour is homeomorphic to a plane.
- Prove that letters C, G, L, and W are all homeomorphic to each other.

2 Holes

Using the rules we agreed upon above, we cannot always transform objects into each other. There are some unchangeable properties, or topological invariants, that allow us to transform cubes into spheres, but do not allow us to transform spheres into donuts. One of such invariants considers a number of holes the object has - if objects have a different number of holes, there is no homeomorphic transformation that will turn object A into object B without breaking the laws of topology. Indeed, if you try to turn a sausage into a donut, you either have to rip it in the middle or connect the far ends, which is not allowed.

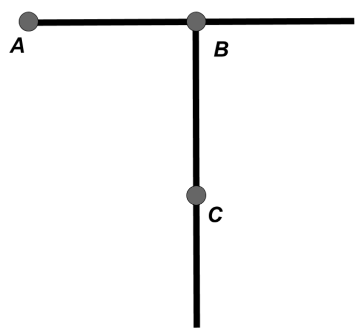
Practice

- Prove that a donut is homeomorphic to a coffee mug.
- Prove that letters A and R are homeomorphic.
- Two objects are isomorphic to each other if they are homeomorphic in a particular "environment", for example, a surface of a cylinder. According to the diagram below, is a meridian of a torus isomorphic to an enclosed curve that traces a triangle on the surface of a torus?
- (SUPER TRICKY!) Prove that a longitude and a median of a torus are isomorphic.



3 Points

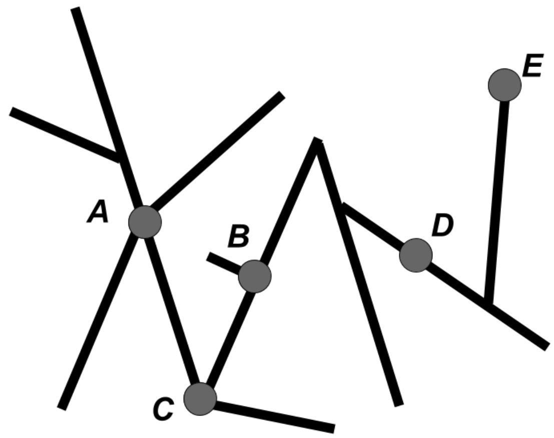
Some points can be taken off from an object, causing it to fall into several pieces. To describe that, every point has an index that states how many lines "fall" or "flow into" that point. Let's take a look at letter T.



Point A has an index of one, point B has an index of three, and point C has an index of two. If we want to prove that one object is or is not homeomorphic to another one, we show that some points on the first object have the same index as or different index from the points on the second object.

Practice

- Find the index of each labeled point on the graph below.



- Prove that letters Q, X, A, and H are not homeomorphic to each other.
- What index do points on a circle have?

4 Problems to think about

- Describe how a punctured bicycle wheel could be turned inside out if it were elastic enough (In reality, it is impossible to do so because the wheels are too stiff)
- What happens when a Klein bottle is cut in half? How many different shapes can one form? many holes do straws have? What about pants?