1 Pascal's Triangle

1.1 How to construct Pascal's Triangle

Begin with a single number: 1. Let this be row 0. In each underlying row, continue by creating numbers that are the sum of the two numbers in the row above it. For example, here is Pascal's Triangle for rows 0 through 6:

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                  1
               1     1
            1     2     1
         1     3     3     1
      1     4     6     4     1
   1     5    10    10     5     1
1     6    15    20    15     6     1
```

Do you notice any patterns?

1.2 Patterns in Pascal's Triangle

The following are exercises to help you explore Pascal's Triangle:

1. To begin, are there any general geometric observations?
2. What do you observe for the diagonal "lines" in the triangle? For example, the third one of these such lines would be: $1, 3, 6, 10, 15 \ldots$
3. What happens if you look at each row as a number? (ex. for row 4, 14641).
   What happens after row 5?
4. What happens adding the numbers in a row?
5. Can you find the Fibonacci Sequence in Pascal's Triangle?

1.3 Sierpinski's Triangle

Notice if you shade the odd numbers, you end up with an interesting figure. This is known as Sierpinski's Triangle, a geometric figure made from repeating equilateral triangles inside other equilateral triangles.

2 Counting and Probability properties in Pascal's Triangle

2.1 Interpretations of Probability

Adopting a system of heads and tails, Pascal's Triangle can show meaningful insights:

1. Using the fact that the sum of the numbers in a row of Pascal's triangle is 2^n, how are the probability spaces of flipping a coin an n number of times represented in Pascal's Triangle? ex. Probability spaces of flipping a coin twice can be HH, TT, HT, TH

2.2 Combinatorics Basics

2.2.1 Definitions

A combination is the number of ways one can choose a certain number of distinguishable objects from a larger group of distinguishable objects. The formula for a combination of choosing r objects from a group of n objects is given as such:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

A combination can be written in the following ways:

$\binom{n}{r}$ $\quad C(n,r)$ $\quad {}_nC_r$ $\quad C_n^r$

2.3 Combinatorical Representation

The following steps will walk you through another insight in Pascal's Triangle.

1. Imagine that Pascal's triangle is now a board game. You can move from one number to another, but only if the next number is both below and adjacent to the previous number.
2. You start from the number 1 in row 0. If your goal is to reach the rightmost number 4 in row 4, how many ways can you reach it?
3. Change your goal to the rightmost 10 in row 5. How many ways can you reach this?
4. Experiment with other ending numbers. Do you see a correlation?

2.4 Binomial Theorem

If we list out what we get from raising (x + y) to various powers we get:

$$
\begin{aligned}
(x+y)^0 &= 1\\
(x+y)^1 &= 1x+1y\\
(x+y)^2 &= 1x^2+2xy+1y^2\\
(x+y)^3 &= 1x^3+3x^2y+3xy^2+1y^3\\
(x+y)^4 &= 1x^4+4x^3y+6x^2y^2+4xy^3+1y^4\\
(x+y)^5 &= 1x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+1y^5
\end{aligned}
$$

We see that the coefficients match up with the numbers of Pascal's triangle.

2.5 Hockey Stick Theorem

To view what the Hockey Stick Theorem is, start from a 1 on the outside of Pascal's Triangle. Now draw a diagonal in a direction, one example of a drawn diagonal is $1- > 3- > 6- > 10$. Then go one unit in the direction of the other diagonal. Using the previous example, the next number reached would be 20. Notice that the sum of the first series of numbers $1+3+6+10$ is equal to the last number 20. Also, if you draw a line through the numbers, it looks like a hockey stick, hence the name of the theorem. The

3 Problems

1. Can you prove the mathematical formula using Pascal's Triangle?:
$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}+\binom{n}{n}=2^n$$
   Can you prove it using the binomial theorem?
2. Dr Steven is trying to create a pyramid out of tennis balls, each with radius 3. He does so by making the first row have one ball, the second row have 3 balls, until he reaches the n^{th} row where he puts the n^{th} triangular number of balls. Find the number of tennis balls in the pyramid if n= 40. (Amador Math Tournament)
3. What is the coefficient of x^4y^6 in the expansion of $(x+y)^{10}$?
4. Take the second element of a row, the third element of a row, and their sum S. What do you notice about the sum of the third element and S?