1 Overview

Bijections are a confusing subject to look at initially, but they can end up being quite useful in different scenarios. Don't be scared by the confusing looking name because we have used bijections many times in the past. To put it simply, a bijection between two sets exists if there is an injection and a surjection between the sets.

2 Injection

What do you think of when you hear injection? You may think of a scary experience with needles in your life, but injections in math are slightly different. An **injection** is when two sets have a one-to-one mapping. Essentially no two elements in one set can map to the same element in another set. Identify the following as injective or not injective. (In the problems below, $\longrightarrow$ represents what something maps to)

1. A mapping from $\{A, B, C, D\}$ to $\{1, 2, 3\}$ satisfying $A \longrightarrow 1$, $B \longrightarrow 2$, $C \longrightarrow 3$, and $D \longrightarrow 3$
2. A mapping from $\{A, B, C\}$ to $\{1, 2, 3\}$ satisfying $A \longrightarrow 1$, $B \longrightarrow 2$, and $C \longrightarrow 3$

3 Surjection

A surjection may sound the same as all of the other jections we have talked about, but it differs slightly. A **surjection** is when all the elements of one set are mapped to by at least one element of another set. Identify the following as surjective or not surjective.

1. A mapping from $\{A, B, C\}$ to $\{1, 2, 3, 4\}$ satisfying $A \longrightarrow 1$, $B \longrightarrow 2$, and $C \longrightarrow 3$
2. A mapping from $\{A, B, C, D\}$ to $\{1, 2, 3, 4\}$ satisfying $A \longrightarrow 4$, $B \longrightarrow 2$, $C \longrightarrow 1$, and $D \longrightarrow 3$

4 Comprehension Check

Now that you know what a surjection and injection is, try to come up with your own examples of the following.

1. The mapping of two sets such that there exists an injection, but NOT a surjection
2. The mapping of two sets such that there exists a surjection, but NOT an injection.
3. The mapping of two sets such that there exists an injection AND a surjection

5 Putting it all together

Remember when we defined a bijection as something that exists when there is both a surjection and an injection? Well now that you know the definitions of injections and surjections, you can create bijections between two objects. Bijections are generally used to make counting problems easier.

1. Recall the problem from last week that consists of someone wanting to travel from the point $(0,0)$ to the point $(7,4)$. Well in this case, we can map moving up to the letter U, and moving right to the letter R. This then becomes a counting problem of how many ways we can rearrange a string which is a problem we know how to solve. Since each string corresponds to a single unique path, it forms a bijection and therefore the answer is correct.

As you can see, you probably have used bijections without even realizing it.

6 Practice Problems

Although these problems may require other technique to be solved, bijections are key in solving this problem. Some of these problems are quite difficult, so don't be discouraged if you get stuck.

1. How many ways are there to pick 5 digit numbers such that no digit is a 0 and the number has digits that are strictly increasing?
2. Recall the concept of stars and bars. Does this concept utilize bijections?
3. How many rectangles can be formed from a 3×3 grid?
4. How many ways are there to place some number of rooks on a chessboard such that no rook can is reached by any other rook? (Note: A rook can only move straight)(HMMT)
5. Challenge: Show that we can count the set of positive rational numbers. (Recall/Note that a set is *countable* if there exists a bijection between the elements of that set and the set of natural, or counting, numbers.)